Higher Order Calculus of Variations (Euler–Poisson equation)

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We can show if we intend to maximize/minimize or technically find a stationary point of the functional S

$$S[y] = \int_a^b L(x, y, y', y'', ..., y^{(n)})dx$$

The solution, given some boundary conditions, will follow this differential equation.

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) + ... + (-1)^n \frac{d^n}{dx^n}\left(\frac{\partial L}{\partial y^{(n)}}\right) = 0$$

This is called the Euler–Poisson equation. Its actually not that much harder to derive this. You can see it has a quite regular and if I dare say beautiful structure.

To start I'm going to attempt to be more formal and define a new function.

$$\phi(\epsilon) = S[f + \epsilon\eta] = S[y]$$

Here eta is our variation. Specifically its a reasonably well behaved function that has as many high order derivatives as our y function. The f function is the min, max, or stationary point we're after.

We also need some constraints to fully define our problem...

$$\eta^{(k)}(a) = \eta^{(k)}(b) = 0 \qquad k \in \{0, ..., n-1\}$$

These are boundary conditions which fix the end points and their first n-1 derivatives. This seems to be the most natural extension of our previous boundary conditions. You are free to try others on your own if you would like however this is outside of my current knowledge.

With that out of the way we can start with the observation that...

$$\left.\frac{d\phi}{d\epsilon}\right|_0 = 0$$

Using Leibniz rule and taking the total derivative we obtain.

$$\delta S = \frac{dS}{d\epsilon} = \int_a^b \left(\frac{\partial L}{\partial y}\frac{dy}{d\epsilon} + \frac{\partial L}{\partial y'}\frac{dy'}{d\epsilon} + ... + \frac{\partial L}{\partial y^{(n)}}\frac{dy^{(n)}}{d\epsilon}\right)dx$$

Now the fun part. We are going to do integral by parts many times. Don't worry though we can do this term by term (Hooray for linearity of integration). Specifically it will suffice to calculate.

$$\int_a^b \frac{\partial L}{\partial y^{(i)}}\frac{dy^{(i)}}{d\epsilon}dx$$

One step of integration by parts along with Leibniz rule gets

$$\frac{\partial L}{\partial y^{(i)}}\eta^{(i-1)}\Big|_a^b - \int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y^{(i)}}\right)\eta^{(i-1)}dx$$

Note that

$$\frac{dy^{(i)}}{d\epsilon} = \eta^{(i)}$$

Using our conditions we get

$$-\int_a^b \frac{d}{dx}\left(\frac{\partial L}{\partial y^{(i)}}\right)\eta^{(i-1)}dx$$

Can you guess a pattern yet? To not bore you with calculus I'm going to skip to the answer. Hopefully its not too hard to see why we get the answer below

$$(-1)^i \int_a^b \frac{d^i}{dx^i}\left(\frac{\partial L}{\partial y^{(i)}}\right)\eta dx$$

So what now? How do we use this information to get our answer? Well we see that all of our terms have an eta hanging out by itself meaning we can rewrite our equation like so

$$\delta S = \int_a^b \eta\left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} + ... + (-1)^n \frac{d^n}{dx^n}\frac{\partial L}{\partial y^{(n)}}\right)dx$$

Were very close; we can literally see our answer under the integral sign. So what's our last step. Well remember that eta is an arbitrary function. We did need it to have some well behaving properties but it is still a very general object. Because of this the only way we can have this integral equal to zero regardless of what we choose eta to be is that the thing in parentheses is also equal to zero.

If you want to be more formal you can look up the fundamental lemma of calculus of variations.

And boom we are done.

Want more generalization madness see

https://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation

under the generalization section.